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TEACHING MATERIAL ON

MATHEMATICS

SCHOOL OF SCIENCE

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Introduction :- (III) Notes on Simplex Method using (i) Two phase (ii) Big-M Methods. (1)

Simplex Method :—

Algorithm :- In certain cases given below, we can't obtain an initial basic feasible solution.

(a) case-I When the constraints are of the type (less than equal to) $\leq$ type

i.e, $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$, $x_j \geq 0$ —(i)

but if some right-hand side constants are negative [i.e, $b_i < 0$] then on adding the non-negative slack variables s_i ($i = 1, 2, \ldots m$), the initial solution is obtained as $s_i = -b_i$ for some i, it is not the feasible solution as non-negativity condition of slack variables is violated. (i.e, $s_i \geq 0$).

(b) case-II When the constraints are of greater than equal to type ($\geq$)

i.e, $\sum_{j=1}^{n} a_{ij} x_j \geq b_i$, $x_j \geq 0$. —(ii)

In this case we add surplus (negative slack) variables to convert the inequalities into equations

i.e. $\sum_{j=1}^{n} a_{ij} x_j - s_i = b_i$, $x_j \geq 0$, $s_i \geq 0$ —(iii)

We assume in this $x_j = 0$ ($\forall j = 1, 2, \ldots n$) to get the initial solution $-s_i = b_i$ or $s_i = -b_i$

We note that here also it is not a feasible solution as it violates the non-negativity ~~Constraints~~ conditions ($s_i \geq 0$) of surplus variable

Hence, we add artificial variables A_i ($i = 1, 2, \ldots, m$) to get an initial basic feasible solution, thus getting the resulting system of equations as :—

$$\left.\begin{aligned} &\sum_{j=1}^{n} a_{ij} x_j - s_i + A_i = b_i \\ &x_j, s_i, A_i \geq 0, \quad i = 1, 2, \ldots m. \end{aligned}\right\} \text{—(iv)}$$

Where m = no. of equations and n = no. of decision variables ~~and~~ so that $(n+m+m)$ are the total no. of variables (i.e, n decision variables, m artificial variables and m surplus variables)

$\Delta_3 = c_3 - c_B Y_3 = 4 - (0, 0, 0)(0, 5, 4) = 4$

If we equate $(n+2m-m) - (n+m)$ variables equal to zero we get an initial basic feasible solution of the new system.

Hence, the new solution of the given LP problem is $A_i = b_i$ $(i=1,2,\cdots,m)$ which does not constitute a solution to the given system of equations (original one) So artificial variable(s) are finally dropped out of the optimal solution.

Thus there are two methods for eliminating these variables from the solution:-

i) Two phase method

ii) Big-M. Method or Method of Penalties.

~~(i) Two phase Method.~~

The solution of problem will itself explain the procedure:-

Ex: Solve the following LP problem using two-phase method:

Minimize $Z = 2x_1 + x_2$

s.t. the constraints:-

$$2x_1 + x_2 \geq 4$$

$$x_1 + 7x_2 \geq 7$$

and $x_1, x_2 \geq 0$

Solution:— Since here in the constraints equation the sign is greater than equal to $(\geq)$. Hence we add surplus variable s_1 and s_2 (-ve slack variables) and artificial variables A_1 and A_2, then the problem becomes

Maximize $Z^*(= -Z) = -x_1 - x_2$. ← { since the problem was of Minimization

Note:- Artificial variables are tools only to generate an initial LP problem solution, which all must be dropped out from the solution mix before the final Simplex solution is reached and this can be achieved by assigning appropriate coefficients to these variables in the objective function and these variables are added to those constraints with equality (=) and greater than or equal to (≥) sign.

1. Two Phase Method:— As the name suggest in the 1st phase the sum of the artificial variables is minimized subject to the given constraints to get a basic feasible solution of the LPP. The second phase minimizes the original objective function starting with the basic feasible solution obtained at the end of the first phase. Since solution of the LPP is obtained in two phases, therefore the method is called as Two phase Method.

Advantages of the Method:—

1. No assumptions on the original system of constraints are made. i.e, the system may have redundant, inconsistent or not solvable in non-negative numbers.
2. In phase-I, an initial basic feasible Solution is easily obtained.
3. The basic feasible Solution (if it exists) obtained at the end of phase-I is used to start phase-II.

STEPS OF ALGORITHM for PHASE-I :—

$\boxed{\text{Step-1}}$ (a) If the given LPP is of minimization then convert it to the maximization type by the usual method.

(b) If all the constraints are of less than or equal to ($\leq$) type then to use directly the phase II to solve the problem; otherwise, the necessary number of surplus and artificial variables are added to convert constraints into equality constraints.

$\boxed{\text{Step-2}}$ We assign zero coefficient to each of the decision variables (x_j) and to the surplus variables and -1 coefficient to each artificial variables are added to convert constraints into equality constraints. This yields an auxiliary LPP. as follows:—

$$\text{Maximize } Z^* = \sum_{i=1}^{m} (-1) A_i$$

s.t constraints

$$\sum_{j=1}^{n} a_{ij} x_j + A_i = b_i, \quad i = 1, 2, \ldots m$$

$$\text{and} \quad x_j, A_j \geq 0$$

$\boxed{\text{Step-3}}$ Now we as usual apply the simplex algorithm to solve this auxiliary LPP problem. The following three cases may arise at optimality.

(i) If Max $Z^* = 0$ and if at least one artificial variable is present in the basis with +ve value then ∄ any feasible soln.

(ii) If Max $Z^* = 0$ and all artificial variables are absent in the basis. Then the basis consists of only decision variables (x_j's) and hence we may move to Phase-II to get an optimal basic feasible soln on the original LPP.

Maximize

(iii) If Max $z^* = 0$ and if at least one artificial variable is present in the basis at zero value. Then a feasible solution to the above LP problem is also a feasible solution to the original LP problem. and we may proceed directly to phase-II for obtaining the basic feasible solution or else eliminate the artificial basic variable and then proceed to phase-II.

Our purpose will be served as and when once an artificial variable has left the basis they can be removed from the simplex table ~~and~~ never to re-enter again into the basis. The value of the objective fn in above problem is bounded below by zero as obj. fn represents the sum of only artificial variables with negative coefficients and thus solution are obtained in a finite no. of steps.

Phase-II :- Assign actual coeffients to the variables in the objective function and zero coefficient to the artificial variables which appear at zero value in the basis at the end of Phase-I. (i.e the last simplex table of Phase I) can be used as the initial simplex table for Phase-II. Then apply the usual simplex algorithm to the modified simplex table to get the optimal solution to the original problem. Artificial variables which do not appear in the basis may be removed.

Question-1 :- Use two-phase simplex method to solve the following LP problem :-

Minimize $Z = 2x_1 + x_2$

s.t. the constraints :-

$$2x_1 + x_2 \geq 4$$
$$x_1 + 7x_2 \geq 7$$
and $x_1, x_2 \geq 0$

Solution :- Since here in each constraints inequation the sign involved is "greater than and equal to" $(\geq)$ therefore we require to add surplus variables S_1, S_2 (i.e, -ve of slack variable) and artificial variables A_1 &

$\Delta_3 = c_3 - c_B Y_3 = 4 - (0,0,0)(0,5,4) = 4$

as such the problem (1) then takes the form :-

Maximize $Z^*(=-Z) = -x_1 - x_2$ [Since the problem was of Minimization type, so we have converted it to Maximization]

S.t. the constraints :

$$\left.\begin{array}{l} 2x_1 + x_2 - S_1 + A_1 = 4 \\ x_1 + 7x_2 - S_2 + A_2 = 7 \\ \text{and } x_1, x_2, S_1, S_2, A_1, A_2 \geq 0 \end{array}\right\} \longrightarrow (1)$$

PHASE-I In phase-I we remove the artificial variables from the basis. Let the equation (1) be the Auxiliary L.P.P. whose initial solution is given by the following table :- (We introduce non-artificial variables at the same level)

Table-I Initial Solution :-

Profit per unit C_B	Basic Variables B	$C_j \rightarrow$ Solution values $b = x_B$	0 x_1	0 x_2	0 S_1	0 S_2	-1 A_1	-1 A_2	Min ratio $x_B/$
-1	A_1	4	2	1	-1	0	1	0	
-1	A_2	7	1	7	0	-1	0	1	
$Z^* = C_B X_B$		Z_j	-3	-8	1	1	-1	-1	
$= -11$		$C_j - Z_j$	3	8 ↑ Max(+ve)	-1	-1	0	0	

In above Eqn (1) we have introduced the surplus variables S_1, S_2 to convert the constraint inequalities to equations and take an

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Also, the given problem is of minimization. So converting it to the maximization by taking the objective function with constraints

$$z^* = -z = -x_1 - x_2$$

$$\left.\begin{aligned} \text{s.t. } & 2x_1 + x_2 - s_1 + A_1 = 4 \\ & x_1 + 7x_2 - s_2 + A_2 = 7 \end{aligned}\right\} \quad \text{--- (1)}$$

and $x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$

<u>Phase-I</u> We assign -1 to artificial variables and cost 0 to all other variables, then new objective function of the auxiliary problem becomes

$$\left.\begin{aligned} \text{Max } z^{**} &= 0\cdot x_1 + 0\cdot x_2 + 0 s_1 + 0\cdot s_2 - 1A_1 - 1A_2 \\ \text{s.t. } & 2x_1 + x_2 - 1s_1 + 0\cdot s_2 + 1A_1 + 0\cdot A_2 = 4 \\ & x_1 + 7x_2 + 0 s_1 - 1\cdot s_2 + 0\cdot A_1 + 1A_2 = 7 \\ \text{And } & x_1, x_2, s_1, s_2, A_1, A_2 \geq 0 \end{aligned}\right\} \text{ (2)}$$

Thus taking $x_1 = 0$, $x_2 = 0$, $s_1 = 0$, $s_2 = 0$ we get the initial basic feasible solution as $A_1 = 4$, $A_2 = 7$. Now we apply the simplex method as usual, we get the following initial solution given by the table as under :- (Note:- We remove the artificial variables from the basis.)

Table-1 Initial Solution

Key element

Key column

		C_j	0	0	0	0	-1	-1	Min Ratio (X_B/x_2)
Profit per unit $(\vec{C_B})$	Basic variable $(\vec{B})$	Solution values $(\vec{b} = X_B)$	x_1	x_2	s_1	s_2	A_1	A_2	(Min. Exchange Ratio)
-1	A_1	4	2	1	-1	0	1	0	$4/1 = 4$
-1	A_2	7	1	7	0	-1	0	1	$7/7 = 1$ (M
$z^{**} = \sum C_B X_B = -11$		z_j	-3	-8	1	1	-1	-1	
		$C_j - z_j$	3	$\uparrow 8$ (most +ve)	-1	-1	0	0	

Δ_1 ... $\Delta_3 = c_3 - c_B Y_3 = 4 - (0, 0, 0)(0, 5, 4) = 4$

We have seen that while maintaining the feasibility of the solution the artificial variables A_1 and A_2 are gradually being removed from the basis. So ~~now~~ moving towards next iteration (N-1) and applying the following row operations in order to get an improved solution by allowing variable 'x_2' to ~~enter~~ enter into the basis and simultaneously first removing variable A_2 from the basis we get the following improved soln. Table-2.

$\boxed{R_2(\text{new}) \rightarrow R_2(\text{old}) / 7 \text{ (Key elt.)}}$ and $\boxed{R_1(\text{new}) \rightarrow R_1(\text{old}) - R_2(\text{new})}$

Table-2 (Improved Soln.)!

Iteration-1

Key Column Key row Key element

Profit per unit (C_B)	Basic variables (B)	Solution values, $b = (X_B)$	$C_j \rightarrow$ 0: x_1	0: x_2	0: s_1	0: s_2	−1: A_1	−1: A_2^*	Minimum exchange Ratio
−1	A_1	$(4-1=)3$	$(2-\frac{1}{7}=)\frac{13}{7}$	0	−1	1/7	1	−1/7 → (outgoing)	$3/(\frac{13}{7}=)\frac{21}{13}$ (Min.)
0	x_2	$(7/7=)1$	1/7	$(7/7=)1$	0	−1/7	0	1/7	$1/(\frac{1}{7})=7$
		Z_j	$-\frac{13}{7}$	0	1	$-\frac{1}{7}$	−1	$\frac{1}{7}$	
$Z^{**} = \sum C_B X_B = -3$		$C_j - Z_j$	$\frac{13}{7}$ ✗ rejected	0	−1	$\frac{1}{7}$ ✓ Accepted (incoming)	0	−8/7	

Note (1): Variable x_1 cannot be allowed to enter into the basis because it would create an infeasible solution.

(2) A_2^* column may be removed forever at this stage.

Now we proceed for 2nd Iteration or Improved solution.

Here in order to remove A_1 from the solution as in above table, we enter variable s_2 in the basis with similar rule of ~~oper~~ row operations (instead of allowing x_1 to enter into the basis which may cause infeasible solution).

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As we proceed for finding feasible solution the A_1, A_2 (artificial variables) gradually being removed from the basis for we follow the following two operations:

Key row: $R_2(\text{new}) \to R_2(\text{old})/7(\text{key element})$ } For Table-1 — Formula

Other row: $R_1(\text{new}) \to R_1(\text{old}) - R_2(\text{new})$

or New value = old value − Corr. Key Column Value × Corr. Key row value

We also allow to enter x_2 variable into the basis & A_2 variable to be removed from the basis simultaneously so, get the following improved (Iterated) Table by similarly repeating the process

Table 2 (Improved soln)

Profit per unit ($\vec{c_B}$)	Basic variables ($\vec{B}$)	Solution values ($\vec{b} = \vec{x_B}$)	x_1 (=a)	x_2 (=b)	s_1 (=c)	s_2 (=d)	A_1	A_2	Minimum Exchange rate or Min. Ratio
			0	0	0	0	−1	−1	
Step-2 −1	A_1	(4−1=) 3	$(2-\frac{1}{7}=)\frac{13}{7}$	0	−1	$\frac{1}{7}$	1	$-\frac{1}{7}$	$(3\div\frac{13}{7}=)\frac{21}{13}$ Minimum (outgoing)
Step-1 0	$b(=x_2)$	$(7/7=)1$	$\frac{1}{7}$	$(\frac{7}{7}=)1$	0	$-\frac{1}{7}$	0	$\frac{1}{7}$	$(1\div\frac{1}{7}=)7$
$z^* = \Sigma c_B x_B = -3$		z_j	$-\frac{13}{7}$	0	1	$-\frac{1}{7}$	−1	$\frac{1}{7}$	
		$c_j - z_j$	$\frac{13}{7}$ (incoming)	0	−1	$\frac{1}{7}$	0	$-\frac{8}{7}$	

We can remove this column

Table 3 (Improve soln)

Profit per unit	Basic variables	Solution values	x_1	x_2	s_1	s_2	
	$a(=x_1)$	$(3\div\frac{13}{7}=)\frac{21}{13}$	1	0	$-\frac{7}{13}$	$\frac{1}{13}$	
0	$b(=x_2)$	$1-\frac{\frac{1}{7}\times 3}{\frac{13}{7}} = \frac{10}{13}$	0	1	$\frac{1}{13}$	$-\frac{2}{13}$	
$z^{**} = c_B x_B = 0$		z_j	0	0	0	0	

In Table 3 (Last table) we find that the value of $z_j \le 0$ $\forall j$, and there is no any artificial variable appearing in the basis, therefore, this solution is an optimal solution to the auxiliary problem.

PHASE – II :- In this phase we assign the actual costs to the original variables, and zero cost to the surplus variables, then the objective fn becomes

$\Delta_3 = c_3 - c_B Y_3 = 4 - (0,0,0)(0,5,4) = 4$

(8)

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Max. $Z^* = -x_1 - x_2 + 0s_1 + 0s_2$ ——— (3)

Now, on replacing the C_j row values by the costs in the above objective function and deleting the artificial values column from the last simplex table in phase-I, we get the first simplex table of phase-II as under:—

Table-4 (Phase-II, Solution)

Profit per unit ($\vec{C_B}$)	Basic variables ($\vec{B}$)	$C_j \rightarrow$ Solution values ($\vec{b} = \vec{X_B}$)	-1 x_1 ($\vec{a}$)	-1 x_2 ($\vec{b}$)	0 s_1 ($\vec{c}$)	0 s_2 ($\vec{d}$)	Minimum Exchange Ratio
-1	(x_1=)a	21/13	1	0	-7/13	1/13	
-1	(x_2=)b	10/13	0	1	1/13	-2/13	
$\therefore Z^* = \sum \vec{C_B}\vec{X_B} = -\frac{31}{13}$		Z_j	0	0	-6/13	-1/13	

Since, the problem was of Maximization type (On Conversion) hence, we ~~see~~ that the value of Z_j is either negative or zero, and meet the optimality condition. ~~[illegible]~~: Hence, the solution is optimal and the solution is given by $x_1 = 21/13$, $x_2 = 10/13$ and

Minimum $Z = -\text{Max } Z^* = -\frac{31}{13}$ Ans.

LPP & SIMPLEX METHOD

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Q1 Example Solve the L.P. problem

Maximize $Z = 3x_1 + 5x_2 + 4x_3$ subject to

$$2x_1 + 3x_2 \le 8$$
$$2x_2 + 5x_3 \le 10$$
$$3x_1 + 2x_2 + 4x_3 \le 15$$

and $x_1, x_2, x_3 \ge 0$.

Soln. Step-1: The problem is a problem of the maximization.

Step-2: All the b's are already +ve

Step-3: Now the inequalities are to be converted to equalities by the introduction of slack variables x_4, x_5, x_6 as follows :—

$$2x_1 + 3x_2 + 1\cdot x_4 + 0\cdot x_5 + 0\cdot x_6 = 8$$
$$0x_1 + 2x_2 + 5\cdot x_3 + 0\cdot x_4 + 1\, x_5 + 0\cdot x_6 = 10$$
$$3x_1 + 2x_2 + 4x_3 + 0\cdot x_4 + 0\cdot x_5 + 1\cdot x_6 = 15$$

Step-4 Taking $x_1 = 0, x_2 = 0, x_3 = 0$ we get $x_4 = 8, x_5 = 10, x_6 = 15$ which is the starting B.F.S.

Step-5: Now we construct starting simplex table :—

Table-1 Starting simplex table:

B	C_B	c_j / $b = X_B$	3 $Y_1 = x_1$ α_1	5 $Y_2 = x_2$ α_2	4 $Y_3 = x_3$ α_3	0 $Y_4 = x_4$ β_1	0 $Y_5 = x_5$ β_2	0 $Y_6 = x_6$ β_3	Min Ratio $\frac{X_B}{Y_2}$
Y_4	0	8	2	$\boxed{3}$	0	1	0	0	$\frac{8}{3}$ (Mini) →
Y_5	0	10	0	2	5	0	1	0	$\frac{10}{2} = 5$
Y_6	0	15	3	2	4	0	0	1	$\frac{15}{2} = 7.5$
$Z = C_B \cdot X_B = 0$		Δ_j	$\Delta_1 = 3$	5	4	0	0	0	

Incoming vector ↑   ↓ outgoing vector

Step-6. Here we compute Δ_j for all zero variables (non-basic) x_j; $J = 1, 2, 3$ by the formula: $\Delta_j = c_j - C_B Y_j$

$\Delta_1 = c_1 - C_B Y_1 = 3 - (0,0,0)(2,0,3) = 3 - \{0 + 0 + 0\} = 3$

$\Delta_2 = c_2 - C_B Y_2 = 5 - (0,0,0)(3,2,2) = 5$

$\Delta_3 = c_3 - C_B Y_3 = 4 - (0,0,0)(0,5,4) = 4$

$\Delta_4 = c_4 - C_B Y_4 = 0 - (0,0,0)(1,0,0) = 0 + 0 + 0$

Obviously $\Delta_4 = 0 = \Delta_5 = \Delta_6$ which corresponds to basic vector Y_i

Step-6 (Contd.): Since all Δ_j's are not less than equal to zero. Therefore, the solution is not optimal solution.

We note that the Δ_j's in the starting simplex table are nothing but c_j's. Thus there is no need to compute them here. So we proceed to the next step.

Step-7 <u>To find the Incoming vector</u> ..

Since $\Delta_2 = 5$ is maximum of $\Delta_1, \Delta_2, \Delta_3$

$\therefore \alpha_2 = (= Y_2)$ is incoming vector.

<u>To find the outgoing vector:-</u>

Since $Y_2 (= \alpha_2)$ is the incoming vector therefore we will consider the ratio $\frac{X_B}{Y_2}$

$$\therefore \frac{X_B}{Y_2} = \left(\frac{X_{B1}}{y_{12}}, \frac{X_{B2}}{y_{22}}, \frac{X_{B3}}{y_{32}}, y_{i2} > 0 \right) = \left(\frac{8}{3}, \frac{10}{2}, \frac{15}{2} \right)$$

$$\text{Since } \frac{X_{Br}}{y_{r2}} = \underset{i}{\text{Mini}} \left[\frac{X_{Bi}}{y_{i2}}, y_{i2} > 0 \right] = \underset{i}{\text{Mini}} \left[\frac{X_{B1}}{y_{12}}, \frac{X_{B2}}{y_{22}}, \frac{X_{B3}}{y_{32}} \right] = \frac{8}{3} = \frac{X_{B1}}{y_{12}}$$

$\therefore$ $r = 1$, i.e., $\beta_1 (= Y_4)$ is the outgoing vector.

Step-8 Since $Y_2 (\alpha_2)$ is incoming vector and $\beta_1 (Y_4)$ is outgoing vector $\therefore$ the key element is $y_{12} (= \alpha_{12}) = 3$ as shown in the table.

$\therefore$ In order to make it equal to 1 we have to divide 1st row by 3 ~~[illegible]~~ and then subtract 2 times of the 1st row (obtained after dividing by 3) from 2nd & 3rd rows and 5 times of the row from the row of Δ_j's to get the new value of Δ_j's.

Thus we construct the second simplex table in which $\beta_1 (= Y_4)$ is now replaced by $\alpha_2 (= Y_2)$ a

Table-2 Second simplex Table.

	c_j	3	5	4	0	0	0	
C_B	X_B	Y_1 $(=\alpha_1)$	Y_2 $(=\beta_1)$	Y_3 $(=\alpha_3)$	Y_4 $(=\alpha_4)$	Y_5 $(=\beta_2)$	Y_6 $(=\beta_3)$	$\frac{X_B}{Y_3}$
Y_2 5	8/3	2/3	1	0	1/3	0	0	$\frac{8/3}{0} = \infty$ ×
Y_5 0	14/3	$-\frac{4}{3}$	0	$\boxed{5}$	$-2/3$	1	0	$\frac{14/3}{5} = \frac{14}{15}$ (Mini) →
Y_6 0	29/3	5/3	0	4	$-2/3$	0	1	$\frac{29/3}{4} = \frac{29}{12}$
$Z = C_B X_B = 40/3$	Δ_j	$-1/3$	0	4	$-5/3$	0	0	

↑ incoming vector ↓ outgoing

Rough:

$10 - 2\,\frac{8}{3} = 10 - \frac{16}{3} = \frac{30-16}{3} = \frac{14}{3}$

$15 - 2\,\frac{8}{3} = 15 - \frac{16}{3} = \frac{45-16}{3} = \frac{29}{3}$

$3 - \frac{4}{3} = \frac{9-4}{3} = \frac{5}{3}$

$3 - \frac{2}{3}\times 5 = 3 - \frac{10}{3} = \frac{9-10}{3} = -1/3$

$\cancel{10} - 5 = -$

$Z = (5,0,0)(8/3, 14/3, 29/3) = \frac{40}{3}$

To check the value of Δ_j we also compute Δ_j's by using the formula

$$\Delta_j = c_j - C_B Y_j \text{ for } x_1, x_3, x_4$$

$\frac{x_{Br}}{y_{r3}} = \underset{i}{\text{Min}}\left[\frac{x_{Bi}}{y_{i3}}, y_{i3} > 0\right] = \underset{i}{\text{Min}}\left[\frac{x_{B3}}{y_{33}}\right]$

Step-9. Since in the 2nd simplex table the values of all Δ_j are not less than or equal to zero therefore, this solution is also not optimal.

Also since $\Delta_j = 4$ is Max^m of all the Δ_j's.

$\therefore \alpha_3 (= Y_3)$ is the incoming vector.

To find outgoing vector :— we find that

$$\frac{x_{Br}}{y_{r3}} = \underset{i}{\text{Mini}}\left[\frac{x_{Bi}}{y_{i3}}, y_{i3} > 0\right] = \underset{i}{\text{Mini}}\left[\frac{x_{B1}}{y_{13}}, \frac{x_{B2}}{y_{23}}, \frac{x_{B3}}{y_{33}}\right]$$

$$= \text{Mini}\left[\infty, \frac{14}{15}, \frac{29}{12}\right] = \frac{14}{15} = \frac{x_{B2}}{y_{23}} \quad \boxed{\therefore r = 2}$$

$\therefore \underline{r=2}$ⁿᵈ, $\beta_2(=Y_5)$ is the outgoing vector and $y_{23} = a_{23} = 5$ is then the key element.

In order to bring $Y_3(\alpha_3)$ in place of ~~Y_5~~ $\beta_2(=Y_5)$ we divide the 2nd row containing the key el... 5 by 5 to get 1 and then subtract 4 times the 2nd row and 4/15 times from the row of Δ_j's.

So the third simplex table will be as follows

3rd simplex Table:-

B	C_B	C_j / X_B	3 / $Y_1(=\alpha_1)$	5 / $Y_2(=\beta_1)$	4 / $Y_3(=\beta_2)$	0 / $Y_4(\alpha_4)$	0 / $Y_5(\alpha_5)$	0 / $Y_6(=\beta_3)$	Min Ratio X_B/Y_1
Y_2	5	8/3	2/3	1	0	1/3	0	0	4
Y_3	4	14/15	-4/15	0	1	-2/15	1/5	0	-7/2 (Neg)
Y_6	0	89/15	41/15	0	0	-2/15	-4/5	1	89/41
$Z=\sum C_B X_B = 256/15$		Δ_j	11/15	0	0	-17/15	-4/5	0	

↑ incoming

Outgoing vector

$\frac{29}{3} - \frac{4 \times 14}{15}$

$\frac{29}{3} - \frac{56}{15} = \frac{145-56}{15}$

$\frac{89}{15}$

Here also we have All Δ_j's are not less than or equal to zero ~~at least~~ $\Delta_1 = 11/15$ Hence the

one more table is to be constructed

Artificial Variable Techniques:- 125

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In L.P.P some constraints may have the sign $\geq$ or $=$ with all b_i's positive. In such problems we introduce surplus variables in the constraints with sign $\geq$. In these problems we cannot get the starting basic matrix $B = I_m$. So to avoid this difficulty we add one more ~~each of ...~~ more variable to each these constraints. As the these variables are called artificial variables. As the names implies these variables are fictitious and present no physical entities. The artificial variable technique is merely a device to get the starting B.F.S so that we may proceed with simplex method to get the optimal solution. Such problems are solved by two methods.

(i) Two phase Method

(ii) Big-M Method or M-technique or Method of penalities due to A. Charnes.

Method 1:- Two Phase Method:-

Here the solution procedure is separated into two phases

Phase-I In phase-I we remove (i.e eliminate) the artificial variables from the basis and introduce other variables (non-artificial variables).

Phase-II we use, the solution of phase-I, as the initial basic solution and apply simplex method to determine the optimum solution.

Computational Procedure of two phase Method:-

Phase-I:-

Step-1 After making all b_i's positive convert each of the constraints into equations by introducing slack, surplus and artificial variables as required.

Step-2 Obtain the new objective function (say Z') by assigning costs -1 to each artificial variable and costs 0 to all other primary, slack and surplus variables.

Thus new objective function

$$Z' = -(\text{Sum of the artificial variables}).$$

Step 3:- Using simplex method maximize the new objective function Z', subject to the constraints of the original problem

If the original problem has a feasible solution then this new problem shall also have an optimal solution with optimal value of the objective function (new objective function z') equal to zero as each of the artificial variables will be equal to zero.

There are three cases that arise:-

(i) If max $z' < 0$ and at least one artificial variable is of the optimal basis at a positive level, then the given problem does not posses any feasible solution, so the procedure is terminated.

(ii) If max $z' = 0$ and at least one artificial variable appears in the optimal basis at zero level, then proceed to phase-II

(iii) If max $z' = 0$ and no artificial variables appears in optimal basis then also proceed to phase-II

It is to be noted that the new objective function z' is always maximization type regardless of whether the given (original) problem is of maximization or minimization type.

Phase-II In phase-II start with the basic feasible solution contained in the final simplex table of phase-I and using simplex method maximize the original objective function z (If original objective fn is of minimization then change it to maximization) with the actual costs of the primal variables and zero costs corresponding to slack and surplus variables.

Ex 6. Solve the following L.P.P by using the two phase method:-

$$\boxed{\begin{aligned} \text{Min } z &= x_1 + x_2 \\ \text{subject to } \; 2x_1 + x_2 &\geq 4 \\ x_1 + 7x_2 &\geq 7 \\ x_1, x_2 &\geq 0 \end{aligned}}$$

Minimize $Z = x_1 + x_2$ subject to $2x_1 + x_2 \geq 4$; $x_1 + 7x_2 \geq 7$ & $x_1, x_2 \geq 0$.

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Solution: The given problem is of minimization. So converting it to maximization by taking the objective function as,

$Z' = -Z = -x_1 - x_2$. and introducing surplus variables x_3, x_4 to convert the constraint inequalities to equations and taking artificial variables x_{a_1} and x_{a_2} the constraint inequalities reduces to the following eqns:-

$$2x_1 + x_2 - x_3 + x_{a_1} = 4$$
$$x_1 + 7x_2 - x_4 + x_{a_2} = 7$$
$$x_1, x_2, x_3, x_4, x_{a_1}, x_{a_2} \geq 0.$$

Phase-I Assigning costs -1 to artificial variables and cost 0 to all other variables, the new objective function of the auxiliary problem becomes.

Maximize $Z'' = 0.x_1 + 0.x_2 + 0.x_3 + 0.x_4 - 1x_{a_1} - 1x_{a_2}$

Subject to the constraints equations given above.

$$2x_1 + x_2 - x_3 + x_{a_1} = 4$$
$$x_1 + 7x_2 - x_4 + x_{a_2} = 7$$
$$x_1, x_2, x_3, x_4, x_{a_1}, x_{a_2} \geq 0$$

$C_B X_B$: $(-1,-1)(4,7)$ $= -4 + (-7) = -11$

Minimum Ratio X_B/Y_2: $\frac{x_{a_1}}{Y_{12}} = \frac{4}{1}$, $\frac{x_{B2}}{y_{22}} = \frac{7}{7} = 1$

Now taking $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0$, we get $x_{a_1} = 4$ & $x_{a_2} = 7$ which is the initial B.F.S. Now applying the simplex method as usual manner, we have the following table:-

Table-1

B	C_B	C_j / X_B	0 / Y_1	0 / Y_2	0 / Y_3	0 / Y_4	-1 / A_1	-1 / A_2	Mini Ratio X_B/Y_2
A_1	-1	4	2	1	-1	0	1	0	4
A_2	-1	7	1	$\boxed{7}$	0	-1	0	1	1 (min) →
$Z'' = C_B X_B = -11$		Δ_j	3	8	-1	-1	0	0	$\frac{X_B}{Y_1}$

↑ Incoming vector; ↓ Outgoing vector

$\Delta_j = C_j - C_B Y_j$

$\Delta_1 = C_1 - C_B Y_1 = 0 - (-1,-1)(2,1) = 0 - \{-2 + (-1)\} = 2+1 = 3$

$\Delta_2 = C_2 - C_B Y_2 = 0 - (-1,-1)(1,7) = 0 - \{-1-7\} = 1+7 = 8$

B	C_B	C_j / X_B	Y_1 (0)	Y_2 (0)	Y_3 (0)	Y_4 (0)	A_1 (−1)	A_2 (−1)	Mini Ratio $\frac{X_B}{Y_1}$
A_1	−1	3	$\boxed{\frac{13}{7}}$	0	−1	$1/7$	1		$\frac{3}{13/7}$
Y_2	0	1	$\frac{1}{7}$	1	0	$-1/7$	0		$\frac{1}{1/7}$
$Z'' = C_B X_B = (-1,0)(3,1) = -3$		Δ_j	$c_1 - C_B Y_1 = 0-(-1,0)(\frac{13}{7},\frac{1}{7}) = \frac{13}{7}$	0	−1	$1/7$	$-1-(-1,0)(1,0) = 0$		

B	C_B	C_j	0	0	0	0	A_1	A_2
~~Y_1~~	~~0~~	~~3~~	~~$\boxed{\frac{13}{7}}$~~	~~0~~	~~−1~~	~~$1/7$~~		
~~Y_2~~	~~0~~	~~1~~	~~$\frac{1}{7}$~~	~~1~~	~~0~~	~~$-1/7$~~		
~~$Z'' = C_B X_B$~~		~~Δ_j~~	~~$\frac{13}{7}$~~	~~0~~	~~−1~~	~~$1/7$~~		

$-\frac{7}{13}$; $1-\frac{9}{13}$; $1-\frac{1}{7} = \frac{7-1}{7} = \frac{6}{7}$; $1-\frac{3}{7}$

B	C_B	X_B	Y_1	Y_2	Y_3	Y_4
Y_1	0	$\frac{21}{13}$	1	0	$-7/13$	$1/13$
Y_2	0	$\frac{10}{13}$	0	1	$1/13$	$-2/13$
$Z'' = C_B X_B$		Δ_j	0	0	0	0

$$\Delta_3 = c_3 - C_B Y_3 = 0 - (0,0)(-7/13, 1/13) = 0 - 0 = 0$$

In the last table all $\Delta_j \leq 0$ and no artificial variables appear in the basis, therefore this solution is an optimal solution to the auxiliary problem.

Phase-II :- Now the 1st simplex table of phase-II is written as follows.

Assigning the actual costs to the original variables and cost zero to the slack, surplus for artificial variables (here surplus variable) the objective function becomes

$$\text{Maximize } Z' = -x_1 - x_2 + 0\cdot x_3 + 0\cdot x_4$$

Now replacing the c_j row values by the costs in the above objective function and deleting the artificial

~~Variable~~ values column from the last simplex table in phase I we write the first simplex table of phase-II as following:—

C_j		-1	-1	0	0	Minimum Ratio
C_B	X_B	Y_1	Y_2	Y_3	Y_4	$X_B/$
-1	$21/13$	1	0	$-7/13$	$1/13$	
-1	$10/13$	0	1	$1/13$	$-2/13$.	
$z = C_B X_B = -31/13$	Δ_j	0	0	$-6/13$	$-1/13$	

$\Delta_1 = c_1 - C_B Y_1 = -1 - (-1,-1)(1,0) = -1+1 = 0$

$\Delta_3 = c_3 - C_B Y_3 = 0 - (-1,-1)(-7/13, 1/13) = \frac{-7}{13} + 1/13 = \frac{-7+1}{13} = \frac{-6}{13}$

$\Delta_4 = c_4 - C_B Y_4 = 0 - (-1,-1)(1/13, -2/13) = 1/13 - 2/13 = \frac{1-2}{13} = -1/13$

Since all $\Delta_j \leq 0$, therefore, this solution is optimal.
Hence the optimal solution of the given problem is

$$x_1 = 21/13, \quad x_2 = 10/13 \text{ and}$$

Computation

$$\text{Minimize } z = -\text{Max } z' = x_1 + x_2$$
$$= \frac{21}{13} + \frac{10}{13} = \frac{31}{13} \text{ Ans}$$

Solve the following LPP by using the two phase Method (11)

Max. $z = 3x_1 + 2x_2 + x_3 + 4x_4$ Subject to the Constraints

$$4x_1 + 5x_2 + x_3 + 5x_4 = 5$$
$$2x_1 - 3x_2 - 4x_3 + 5x_4 = 7$$
$$x_1 + 4x_2 + 5x_3 - 4x_4 = 6$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Sol. Introducing artificial variables $x_{a_1}, x_{a_2}, x_{a_3}$, the constraints reduces to

$$4x_1 + 5x_2 + x_3 + 5x_4 + x_{a_1} = 5$$
$$2x_1 - 3x_2 - 4x_3 + 5x_4 + x_{a_2} = 7$$
$$x_1 + 4x_2 + 5x_3 - 4x_4 + x_{a_3} = 6$$
$$x_1, x_2, x_3, x_4, x_{a_1}, x_{a_2}, x_{a_3} \geq 0$$

Phase-I Assigning cost (-1) to artificial variables and cost 0 to all other variables, the objective function becomes

$$\text{Max } Z' = 0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 + 0\cdot x_4 - x_{a_1} - x_{a_2} - x_{a_3}$$

subject to the constraints given above.

Now taking $x_1 = x_2 = x_3 = x_4 = 0$ we get $x_{a_1} = 5$, $x_{a_2} = 7$, $x_{a_3} = 6$, which is the starting basic feasible solution.

Now applying the simplex method to this auxiliary problem, in the usual manner, we have the following table :—

B	C_B	C_j / X_B	0 / Y_1	0 / Y_2	0 / Y_3	0 / Y_4	-1 / A_1	-1 / A_2	-1 / A_3	Mini Ratio
A_1	-1	5	$\boxed{4}$	5	1	5	1	0	0	X_B/Y_1: $\frac{X_{B_1}}{Y_{11}} = \frac{5}{4}$ (Min)
A_2	-1	7	2	-3	-4	5	0	1	0	$\frac{X_{B_2}}{Y_{21}} = 7/2$
A_3	-1	6	1	4	5	-4	0	0	1	$\frac{X_{B_3}}{Y_{31}} = \frac{6}{1}$
$Z' = C_B' X_B = -18$		Δ_j	7 (incoming vector)	6	2	6	0 (outgoing vector)	0	0	X_B/Y_3
Y_1	0	5/4	1	5/4	1/4	5/4	—	0	0	$\frac{X_{B_1}}{Y_{13}} = \frac{5/4}{1/4} = 5$
A_2	-1	9/2	0	-11/2	-9/2	5/2	—	1	0	$\frac{X_{B_2}}{Y_{23}} = \frac{9/2}{-9/2} = -1$ (negative)
A_3	-1	19/4	0	1/4	$\boxed{19/4}$	$-\frac{21}{4}$	—	0	1	$\frac{X_{B_3}}{Y_{33}} = \frac{19/4}{19/4} = 1$ (Min)
		Δ_j	0	-11/4	1/4 (Incoming vector)	-11/4		0 (Outgoing vector)	0	$X_B/$
Y_1	0	1	1	0	0	29/19	—	0	—	
A_2	-1	9	0	-55/19	0	-47/19	—	1	—	—
Y_3	0	1	0	11/19	1	-21/19	—	0	—	
$Z' = C_B X_B$		Δ_j	0	-55/19	0	-47/19		0		

Since all $\Delta_j \le 0$ therefore an optimal B.F.S to the auxiliary problem has been attained. 65

But the artificial variable vector (corresponding to x_{a_2} appears in the ~~basis~~ optimal basic (basic solution) or in basis at a positive level. Hence, the auxiliary as well as the original L.P.P does not possess any feasible solution.

Examples:- 14 (i) ~~/364~~ 15 (iv) ~~365~~,

Art 9.11/387, Ex 8/392

Art $\frac{21.7}{791}$ Ex 8/797, E. 12/804

Q 14 (i) Max $Z = 2x_1 + x_2$

s.t : $3x_1 + x_2 = 3$

$4x_1 + 3x_2 \ge 6$

$x_1 + 2x_2 \le 4$

$x_1, x_2 \ge 0$.

Sol: Examining the constraints carefully we note that ~~the~~ ~~order to~~ constraints equations contains $=$, $\ge$ & $\le$ sign. Therefore introducing slack, surplus and artificial variables we get,

~~$\text{Max } Z = 2x_1 + x_2$~~

s.t. $3x_1 + x_2 + 0.x_3 + 0x_4 + x_{a_1} = 3$.

$4x_1 + 3x_2 - x_3 + 0x_4 + x_{a_2} = 6$

$x_1 + 2x_2 + 0.x_3 + x_4 = 4$

$\text{Max } Z = 2x_1 + x_2 + 0x_3 - 0x_4 - Mx_{a_1} - Mx_{a_2}$

$3x_1 + x_2 + 0.x_3 + 0.x_4 + x_{a_1} = 3$

$4x_1 + 3x_2 - x_3 + 0.x_4 + \quad + x_{a_2} = 6$

$x_1 + 2x_2 + 0x_3 + x_4 \quad = 4$

$x_1, x_2 \ge 0$

Qn. Max $Z = -2x_1 - x_2$

s.t. $3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 4$

$x_1, x_2 \geq 0$.

Soln. Step-1 Introducing slack surplus and artificial variables, the system of constraints equation becomes:-

$$3x_1 + x_2 + 0x_3 + 0 \cdot x_4 + xa_1 + 0xa_2 = 3$$

Surplus $4x_1 + 3x_2 - x_3 + 0 \cdot x_4 + 0xa_1 + xa_2 = 6$

Slack $x_1 + 2x_2 + 0 \cdot x_3 + x_4 + 0 \cdot xa_1 + 0 \cdot xa_2 = 4$

which can be written in Matrix form as.

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & A_1 & A_2 \end{array}$$
$$\begin{bmatrix} 3 & 1 & 0 & 0 & 1 & 0 \\ 4 & 3 & -1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ xa_1 \\ xa_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix}$$

Step 2. Assigning the large negative price $-M$ to the artificial variables xa_1 and xa_2, the objective function becomes:-

$$\text{Max } Z = -2x_1 - x_2 + 0x_3 + 0 \cdot x_4 - M xa_1 - M xa_2$$

Step-3. Now constructing starting simplex table as follows:-

Table (133)

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C_B	C_j / X_B	$+2$ / Y_1	$+1$ / Y_2	0 / Y_3	0 / Y_4	$-M$ / A_1	$-M$ / A_2	Min Ratio $\frac{X_B}{Y_1}$
$-M$	3	$\boxed{3}$	1	0	0	1	0	$\frac{X_{B1}}{y_{11}} = \frac{3}{3} = 1$ (Min) →
$-M$	6	4	3	-1	0	0	1	$\frac{X_{B2}}{y_{21}} = \frac{6}{4} = \frac{3}{2}$
0	4	1	2	0	1	0	0	$\frac{X_{B3}}{y_{31}} = \frac{4}{1} = 4$
$Z = C_B X_B = -9M$	Δ_j	$2+7M$ ↑ incoming vector	$1-4M$	M	0	0 ↓ outgoing vector	0	X_B/Y_2

A_1 has been eliminated forming the next tableau

C_B	X_B	Y_1	Y_2	Y_3	Y_4	A_1	A_2	X_B/Y_2
2	1	1	$1/3$	0	0		0	3
$-M$	2	0	$\boxed{5/3}$	-1	0		1	$6/5$ (Min) →
0	3	0	$5/3$	0	1		0	$9/5$
$Z = C_B X_B = 2-2M$	Δ_j	0	$(1-5M)/3$ ↑ incoming	M	0		0 ↓ outgoing	

C_B	X_B	Y_1	Y_2	Y_3	Y_4
2	$3/5$	1	0	$-1/5$	0
1	$6/5$	0	1	$-3/5$	0
0	1	0	0	[illegible]	1
$Z = C_B X_B$	Δ_j	0	0	$1/5$	0

~~Thus $x_1 =$~~ ∴ Clearly here all Δ_j are positive and consequently, the optimum sol. is

$x_1 = 3/5$, $x_2 = 6/5$ ~~and~~ $Z(\text{Max}) = 12/5$

Ans

15 (IV) Using two phase simplex method to solve the problem:

Minimize $z = x_1 - 2x_2 - 3x_3$ Subject to the constraints

$-2x_1 + x_2 + 3x_3 = 2$

$2x_1 + 3x_2 + 4x_3 = 1,$

$x_1, x_2, x_3 \geq 0$

Soln:- Let us convert the objective function from Minimization problem to maximization by writing

Maximize $z' = -z = -x + 2x_2 + 3x_3$ subject to the constraints

$-2x_1 + x_2 + 3x_3 + xa_1 + 0.xa_2 = 2$

$2x_1 + 3x_2 + 4x_3 + 0.xa_1 + 1.xa_2 = 1$

$x_1, x_2, x_3, xa_1, xa_2 \geq 0.$

Phase-1 Auxiliary Linear Programming problem is

Max $z'^* = 0x_1 + 0x_2 + 0x_3 - 1xa_1 - 1xa_2$ subject to the above constraints. The following solution table is obtained:-

Table-1

Variables B	C_B	C_j / X_B	0 / Y_1	0 / Y_2	0 / Y_3	-1 / A_1	-1 / A_2	Min Ratio $X_B/$
A_1	-1	2	-2	1	3	1	0	
A_2	-1	1	2	3	4	0	1	
$z^* = C_B X_B = -3$		Δj	0	-4	-7	0	0	

Since all $\Delta j \geq 0$ an optimal basic feasible solution to the auxiliary LPP has been obtained. But at the same time z^* is negative and the artificial variable x_{a_1} appears in the basic solution at a positive level. Hence the problem does not possess any feasible solution. Hence Phase II is omitted.

Duality in Linear Programming ~~X0 NEED TO STUDY~~ 17

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9.11. Qn. What is Dual Simplex Method? What is the advantage. Write computational procedure of the dual simplex Algorithm.

1: To form the given L.P.P. in standard primal form:—

(i) If the problem is of minimization. Convert it into the maximization problem.

(ii) Write all the constraints in the form of inequalities involving $\leq$ sign

Note that by doing so some b_i's may change to negative values.

Step-II To find initial basic solution:—

1. Introduce the slack variables in the constraints to reduce them to equalities.

2. Find the initial basic solution taking all the given variables equal to zero and calculate the values of the slack variables. This solution will be the starting (initial) basic solution, which may not be feasible.

3. Let $X_B = (x_{B_1}, x_{B_2}, x_{B_3}, \ldots, x_{B_m})$ be the initial basic solution corresponding to the basis matrix ~~B~~ $B = (\beta_1, \beta_2, \ldots, \beta_m)$

Step-III: Construction of the starting simplex table:—

Construct the simplex table as usual in simplex method

Step-IV: To test the initial solution for optimality:—

We compute $\Delta_j = c_j - C_B Y_j$ for every column.

(i) If all $\Delta_j \leq 0$ and all b_i are non-negative, the solution found above is an optimal basic feasible solution

(ii) If all $\Delta_j \leq 0$ and at least one b_i is negative, then proceed to ~~steps~~ steps (V)

(iii) If any $\Delta_j > 0$ then the method fails.

** To add Dummy

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Step-V To find the vector coming (entering) and leaving (outgoing) the basis :→

i) To determine the out going vector :– Here β_r i.e, the r-th column in the basis i.e, the corresponding vector $\bar{x}_r$ in the basis is the outgoing vector of

$$x_{Br} = \text{Min}\{x_{Bi} ;\ x_{Bi} < 0\}$$

(ii) To determine the incoming vector (α_k) :→ If β_r is the outgoing vector, then α_k is taken as the entering (incoming) vector for the value of k, for which

$$\frac{\Delta_k}{y_{rk}} = \underset{j}{\text{Min}}\left[\frac{\Delta_j}{y_{rj}},\ y_{rj} < 0\right]$$

If all $y_{rj} \geq 0$, then the problem has no F.S.

Step VI :– Test of optimality :→ If entering the vector α_q in place of β_r in the basis all basic variables reduces to non-negative values, then this sol^n is optimal F.S. But if at least one basic variable is negative, then this solution is not optimal F.S. In this case repeat IV and V, iteratively till an optimal F.S is obtained.

Ex 8 392 Use dual simplex method to solve the following L.P.P :—

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Minimize $Z = 6x_1 + 7x_2 + 3x_3 + 5x_4$.

Subject to $5x_1 + 6x_2 - 3x_3 + 4x_4 \geq 12$

$x_2 + 5x_3 - 6x_4 \geq 10$

$2x_1 + 5x_2 + x_3 + x_4 \geq 8$

and $x_1, x_2, x_3, x_4 \geq 0$.

Step I The given L.P.P in the standard primal form is given by

$$\text{Max } z_p = -Z = -6x_1 - 7x_2 - 3x_3 - 5x_4$$

s.t. $-5x_1 - 6x_2 + 3x_3 - 4x_4 \leq -12$

$-x_2 - 5x_3 + 6x_4 \leq -10$

$-2x_1 - 5x_2 - x_3 - x_4 \leq -8$. and $x_1, x_2, x_3, x_4 \geq 0$

Since objective function is of maximization and all $c_j < 0$, we can solve this L.P.P by dual simplex algorithm :—

Step II Introducing the slack variables x_5, x_6 and x_7 the constraints of the above problem reduce to the following equalities.

$$-5x_1 - 6x_2 + 3x_3 - 4x_4 + x_5 + 0.x_6 + 0.x_7 = -12$$

$$-0.x_1 - x_2 - 5x_3 + 6x_4 + 0.x_5 + x_6 + 0.x_7 = -10$$

$$-2x_1 - 5x_2 - x_3 - x_4 + 0.x_5 + 0.x_6 + x_7 = -8$$

$x_1, x_2, x_3, x_4, x_6, x_7, x_8 \geq 0$

Now taking $x_1 = x_2 = x_3 = x_4 = 0$ we have.

$x_5 = -12, x_6 = -10, x_7 = -8$ which is the starting basic solution to the primal and is infeasible.

Step III we construct the starting simplex table as follows :—

Table 1 (138)

B	C_B	c_j: / x_B	$Y_1(x_1)$ -6	$Y_2(x_2)$ -7	$Y_3(x_3)$ -3	$Y_4(x_4)$ -5	$Y_5(\beta_1)$ 0	$Y_6(\beta_2)$ 0	$Y_7(\beta_3)$ 0	Mini Ratio
Y_5	0	-12	-5	$\boxed{-6}$	3	-4	1	0	0	→
Y_6	0	-10	0	-1	-5	6	0	1	0	
Y_7	0	-8	-2	-5	-1	-1	0	0	1	
$Z_B = 0$		Δ_j:	-6	-7	-3	-5	0	0	0	
				↑			↓			

Step IV. $\Delta_1 = c_1 - z_1 = c_1 - C_B Y_1 = -6$,

$\Delta_2 = -7,\ \Delta_3 = -3,\ \Delta_4 = -5,\ \Delta_5 = 0 = \Delta_6 = \Delta_7$.

Thus starting basic solution is infeasible but optimal.

** Different

Step-V: (i) To determine the leaving vector (β_r)

since $\boxed{x_{\beta_r} = \text{Min}(x_{B_i}, x_{B_i} < 0)} = \text{Min}(-12, -10, -8)$

$= -12 = x_{B_1}$

formula

$\therefore \boxed{r = 1}$ i.e. $(\beta_1 = Y_5)$ is the leaving vector

(ii) To determine the outgoing vector (α_k)

$$\boxed{\frac{\Delta_k}{y_{rk}} = \frac{\Delta_k}{y_{1k}} = \underset{j}{\text{Mini}}\left\{\frac{\Delta_j}{y_{1j}} < 0\right\}} = \text{Mini}\left\{\frac{\Delta_1}{y_{11}}, \frac{\Delta_2}{y_{12}}, \frac{\Delta_4}{y_{14}}\right\}$$

formula

$$= \text{Mini}\left\{\frac{-6}{-5}, \frac{-7}{-6}, \frac{-5}{-4}\right\} = \frac{7}{6} = \frac{\Delta_2}{y_{12}}$$

$\therefore K = 2$ i.e. $\alpha_2 (= Y_2)$ is the entering vector

$\therefore$ key element is $y_{12} = -6$

Proceeding as usual the second simplex table is as follows:—

Table-2

C_B	c_j / X_B	-6 / Y_1	-7 / $Y_2(\beta_1)$	-3 / Y_3	-5 / Y_4	0 / Y_5	0 / $Y_6(\beta_2)$	0 / $Y_7(\beta_3)$
-7	2	5/6	1	-1/2	2/3	-1/6	0	0
0	-8	5/6	0	$\boxed{-11/2}$	2/3	-1/6	1	0 →
0	2	13/6	0	-7/2	7/3	-5/6	0	1
z = -14	Δ_j	-1/6	0	-13/2 ↑	-1/3	-7/6	0 ↓	0

The solution given in this table is $x_1 = 0 = x_3 = x_4 = x_5$, $x_2 = 2$, $x_6 = -8$, $x_7 = 2$, which is infeasible and optimal, which can be improved further.

To determine the leaving vector (β_r)

Since $x_{Br} = \underset{i}{\text{Min}}\,(x_{Bi},\ x_{Bi} < 0) = \text{Min}\,(x_{B2})$

$= \text{Min}\,(-8) = -8 = x_{B2}$ $\boxed{\therefore r = 2}$

$\therefore$ i.e. $\beta_2\ (= Y_6)$ is the leaving vector.

To determine the entering vector (α_k) :—

$$\frac{\Delta_k}{y_{rk}} = \frac{\Delta_k}{y_{2k}}\ (\because r=2) = \underset{j}{\text{Mini}}\left\{\frac{\Delta_j}{y_{2j}},\ y_{2j} < 0\right\} = \text{Mini}\left\{\frac{\Delta_3}{y_{23}}, \frac{\Delta_5}{y_{25}}\right\}$$

$$= \text{Mini}\left\{\frac{-13/2}{-11/2}, \frac{-7/6}{-1/6}\right\} = \text{Mini}\left\{\frac{13}{11}, \frac{7}{1}\right\} = \frac{13}{11} = \frac{\Delta_3}{y_{23}}$$

$k = 3$ i.e. $\alpha_3\ (= Y_3)$ is the entering vector.

$\therefore$ Key element $= y_{23} = -11/2$

Proceeding as usual the third simplex table is as follows :—

Table-3

C_B	c_j / X_B	-6 / Y_1	-7 / Y_2	-3 / Y_3	-5 / Y_4	0 / Y_5	0 / Y_6	0 / Y_7
-7	30/11	25/33	1	0	2/33	-5/33	-1/11	0
-3	16/11	-5/33	0	1	-40/33	1/33	-2/11	0
0	78/11	18/11	0	0	-21/11	-8/11	-7/11	1
-258/11	Δ_j	-38/33	0	0	-271/33	-32/33	-13/11	0

The Solution given in this Table is (140) (22)

$x_1 = 0 = x_4 = x_5 = x_8$, $x_2 = 30/11$, $x_3 = 16/11$ and $x_7 = 78/11$, which is feasible and optimal.

Hence, the optimal feasible solution of the given LPP is $x_1 = 0$, $x_2 = 30/11$, $x_3 = 16/11$, $x_4 = 0$ and Min $z = -$ Max $z_p = 258/11$ Ans.

(7)

New Chapter — NON-Linear Programming Problem

§ 21.7 / 791 — Kuhn-Tucker Necessary conditions for the Optimality of the Objective Function in a GNLP ~~Prof.~~ Problem :—

Soln: Let us consider the General Non-linear Programming Problem (GNLP) :—

Maximize $z = f(\bar{x})$, $\bar{x} = (x_1, x_2, \ldots, x_n) \in R^n$

Subject to the constraints :—

$$g_i(\bar{x}) \leq b_i, \quad i = 1, 2, \ldots, m \ (m < n)$$

where b_i's are constants.

Let $h_i(\bar{x})$ be, m function, such that $h_i(\bar{x}) = g_i(\bar{x}) - b_i$ $\forall\ i = 1, 2 \ldots m$, then the constraints $g_i(\bar{x}) \leq b_i$ of the problem may be replaced by the constraints,

$$h_i(\bar{x}) = g_i(\bar{x}) - b_i \leq 0 \quad \forall\ i = 1, 2, \ldots m.$$

Now introducing the slack variables i.e., $s_1, s_2, \ldots, s_m$, these inequality constraints reduce to the equalities as

$$h_i(\bar{x}) + s_i^2 = 0, \quad i = 1, 2, \ldots, m$$

Here s_i^2 have been added to ensure that the quantity added to $h_i(\bar{x})$, to make it the equality, is non-negative, for all $i = 1, 2, \ldots, m$. By doing so m extra restrictions, that $s_i \geq 0$ are also avoided.

Thus the given NLPP reduces to the following NLPP :—

Optimize $z = f(\bar{x})$, $\bar{x} = (x_1, x_2, \dots x_n) \in R^n$. 14.1

Subject to the constraints

$$h_i(\bar{x}) + s_i^2 = 0 \quad \forall\, i = 1, 2, \dots, m$$

and $\bar{x} \geq 0$ i.e., $x_j \geq 0$, $\forall\, j = 1, 2, \dots, n$.

which is a NLPP in $n+m$ variables x_j, s_i, $i = 1, 2, \dots, m$; $j = 1, 2, \dots n$.

with the m equality constraints and thus it can be solved by using the Lagrangian multipliers method:

Introducing the Lagrangian multipliers $\lambda_1, \lambda_2, \dots, \lambda_m$ the Lagrangian function is defined as follows:—

$$L(\bar{x}, s, \lambda) = f(\bar{x}) - \sum_{i=1}^{m} \lambda_i [h_i(\bar{x}) + s_i^2]$$

$\bar{x} = (x_1, x_2, \dots, x_n)$, $s = (s_1, s_2, \dots, s_m)$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$

Assuming that L, f and h_i are all differentiable partially with respect to $x_1, x_2, \dots, x_n$; $\lambda_1, \lambda_2, \dots, \lambda_m$; $s_1, s_2, \dots, s_m$, the necessary conditions for the stationary points are

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial h_i}{\partial x_j} = 0 \text{ for all } j = 1, 2, \dots n. \quad (1)$$

$$\frac{\partial L}{\partial \lambda_i} = -[h_i(\bar{x}) + s_i^2] = 0, \quad i = 1, 2, \dots m. \quad (2)$$

&

$$\frac{\partial L}{\partial s_i} = -2 s_i \lambda_i = 0, \quad i = 1, 2, \dots, m \quad (3)$$

From (3), we have either $s_i = 0$ or $\lambda_i = 0$

If $s_i = 0$, then (2) $\Rightarrow h_i(\bar{x}) = 0$ $\therefore \lambda_i = 0$ or $s_i = 0 \Rightarrow \lambda_i = 0$ or $h_i(\bar{x}) = 0$. Thus, we may write $\lambda_i h_i(\bar{x}) = 0$

$\because s_i^2 \geq 0$, so eqn (2) $\Rightarrow h_i(\bar{x}) \leq 0$

When $h_i(\bar{x}) < 0$, (4) $\Rightarrow \lambda_i = 0$ and when λ_i is unrestricted in sign.

If $\lambda_i \neq 0$ then $s_i = 0$, in that case from (2), we have

$$h_i(\bar{x}) = g_i(\bar{x}) - b_i = 0 \text{ i.e., } g_i(\bar{x}) = b_i$$

Since λ_i represent the rate of change of f w.r.t. b_i

$$\frac{\partial f}{\partial b_i} = \lambda_i$$

Now as the right hand side of $h_i(\bar{x}) \le 0$ (142) ③
increases above zero, the solution space becomes less constraint ⇒ f cannot decreases,
so $\frac{\partial f}{\partial b_i} = \lambda_i \nless 0$. i.e, $\lambda_i \ge 0$

Thus when $h(\bar{x}) < 0$, $\lambda = 0$ and when $\lambda > 0$, $h(\bar{x}) =$

Hence, the Kuhn-Tucker necessary conditions for the point $\bar{x}$ to be a point of maximum for $f(\bar{x})$, subject to $h_i(\bar{x}) = g_i(\bar{x}) - b_i \le 0$ are as follows :—

$$\frac{\partial f(\bar{x})}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial h_i(\bar{x})}{\partial x_j} = 0 \text{ for all } j = 1, 2, \cdots, n$$
$$\lambda_i h_i(\bar{x}) = 0, \quad h_i(\bar{x}) \le 0 \text{ and } \lambda_i \ge 0, \quad i = 1, 2, \cdots, m.$$

Similarly the Kuhn-Tucker necessary conditions for the point $\bar{x}$ to be a point of minimum for $f(\bar{x})$, subject to $h_i(\bar{x}) = g_i(\bar{x}) - b_i \ge 0$ are as follows: —

$$\frac{\partial f(\bar{x})}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial h_i(\bar{x})}{\partial x_j} = 0, \text{ for all } j = 1, 2, \cdots, n$$
$$\lambda_i h_i(\bar{x}) = 0, \quad h_i(\bar{x}) \ge 0 \text{ and } \lambda_i \ge 0, \quad i = 1, 2, \cdots, m.$$

§ 21.8. 793. **Kuhn-Tucker Sufficient Conditions for the optimality of the Objective Function of a GNLPP with Inequality Constraints.**

To solve problem

1. The Kuhn-Tucker necessary condition for NLPP, maximize $f(\bar{x})$, $\bar{x} \in R^n$ subject to the constraints $h_i(\bar{x}) = g_i(\bar{x}) - b_i \le 0$ $(i = 1, 2, \cdots, m)$, $\bar{x} \ge 0$ are also the sufficient conditions for maximum of $f(\bar{x})$, if (i) $f(x)$ is concave and (ii) $h_i(\bar{x})$ [i.e, $g_i(\bar{x})$] are convex function of $\bar{x}$, i.e, $-h_i(\bar{x})$ (i.e, $g_i(x)$) are also concave functions of $\bar{x}$ for all $i = 1, 2, \cdots, m$

The Kuhn-Tucker necessary conditions for NLPP maximize $f(\bar{x})$, $x \in R^n$, subject to the constraints

$h_i(\bar{x}) = g_i(\bar{x}) - b_i \le 0, \bar{x} \ge 0$ ④

are $\frac{\partial f(\bar{x})}{\partial x_j} = \sum_{i=1}^{m} \lambda_i \frac{\partial h_i(\bar{x})}{\partial x_j}$, for all $j = 1, 2, \dots, n$. 7]

$\lambda_i h_i(\bar{x}) = 0, h_i(\bar{x}) \ge 0$ and $\lambda_i \ge 0, i = 1, 2, \dots, m$.

The Lagrangian function of the problem can be written as $L(\bar{x}, s, \lambda) = f(\bar{x}) - \sum_{i=1}^{m} \lambda_i [h_i(\bar{x}) + s_i^2]$

where $\bar{x} = (x_1, x_2, \dots, x_n)$, $s = (s_1, s_2, \dots, s_m)$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$

$s_1, s_2, \dots s_n$ are slack variables ~~and~~ used to convert inequality constraints to equality constraints, and $\lambda_1, \lambda_2, \dots, \lambda_m$ are the Lagrangian multipliers. — 1/2

Now we proceed to prove that $L(\bar{x}, s, \lambda)$ as defined above is concave function.

Using the slack variables s_i, the inequality constraints of the problem reduces to the equality constraints $h_i(\bar{x}) + s_i^2 = 0, i = 1, 2, \dots, m$. and from the necessary conditions, we have $\lambda_i h_i(\bar{x}) = 0$, $i = 1, 2, \dots, m$ $\therefore \lambda_i s_i^2 = -\lambda_i h_i(\bar{x}) = 0, i = 1, 2, \dots, m$.

Since $h_i(\bar{x})$ are convex function of $\bar{x}$ and $\lambda_i \ge 0$, it follows that $\lambda_i h_i(\bar{x})$ is convex function

$\Rightarrow -\lambda_i h_i(\bar{x})$ is concave function.

$\Rightarrow -\sum_{i=1}^{m} \lambda_i h_i(\bar{x})$ is concave function of $\bar{x}$.

$\therefore f(\bar{x}) - \sum_{i=1}^{m} \lambda_i h_i(\bar{x})$ is concave function of $\bar{x}$

$\Rightarrow f(\bar{x}) - \sum_{i=1}^{m} \lambda_i [h_i(\bar{x}) + s_i^2]$ is concave fn $\because \lambda_i s_i^2 = 0$

$\Rightarrow L(\bar{x}, s, \lambda)$ is concave function.

The necessary condition for maximum of $f(\bar{x})$, at an extreme point; implies that $L(\bar{x}, s, \lambda)$ also have the same extreme point. Since $L(\bar{x}, s, \lambda)$ is concave, its derivative must be zero at one point and obviously this point must give the absolute max^m value of $f(\bar{x})$.

2. The Kuhn - Tucker necessary condition for the minimization NLPP :- 144 (5)

Minimize $f(\bar{x})$, subject to the constraints

$$h_i(\bar{x}) = g_i(\bar{x}) - b_i \geq 0, \; i = 1, 2, \ldots, m \text{ and } \bar{x} \geq 0;$$

$\bar{x} = (x_1, x_2, \ldots, x_n) \in R^n$ are also the sufficient conditions for minimum of $f(\bar{x})$ if

(i) $f(\bar{x})$ is convex and

(ii) $h_i(\bar{x})$ (i.e., $g_i(\bar{x})$) are also convex function of $\bar{x}$, for all $i = 1, 2, \ldots, m$.

Proof :- Need to proceed similarly left as an exercise for the reader.

Tips :- (1) In general NLPP may contain the constraints with $\geq$ or $=$ or $\leq$ sign. All the constraints must be converted to the type $\leq$ in the case of maximization of NLPP and to the type $\geq$ in the case of minimization of NLPP. The inequality of type $\leq$ may be converted into inequality of type $\geq$, by multiplying both sides of the inequality by -1 and vice-versa.

(2) In both maximization or minimization of NLPP the Lagrangian multiplier λ_i corresponds to the equality constraints $h_i(\bar{x}) = 0$ must be unrestricted in sign.

Tips :- (1) Since $h_i(\bar{x}) = g_i(\bar{x}) - b_i$, where b_i is constant, so $h_i(\bar{x})$ is concave or convex $\Rightarrow g_i(\bar{x})$ is concave or convex respectively.

(2) If in the minimizing NLPP the constraints are taken of the type $h_i(\bar{x}) \leq 0$, then the Kuhn-Tucker necessary condition for the optimality of the objective fn will be

$$\frac{\partial f(\bar{x})}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial h_i(\bar{x})}{\partial x_j} = 0, \; \forall j = 1, 2, \ldots, n$$

$$\lambda_i h_i(\bar{x}) = 0, \; h_i(\bar{x}) \leq 0 \text{ and } \lambda_i \leq 0 \; \forall i = 1, 2, \ldots, m$$

and these conditions will be sufficient if $f(\bar{x})$ and all $h_i(\bar{x})$ are convex for $\bar{x}$ ...

88. Determine x_1, x_2, x_3 to maximize

$$Z = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$$

Subject to the constraints:

$$x_1 + x_2 \leq 2, \quad 2x_1 + 3x_2 \leq 12 \text{ and } x_1, x_2 \geq 0$$

Soln. Here $f(x) = Z = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$,

$x = (x_1, x_2, x_3) \in R^3$

$$\boxed{\begin{aligned} h_1(x) &= x_1 + x_2 - 2 \leq 0 \\ h_2(x) &= 2x_1 + 3x_2 - 12 \leq 0 \end{aligned}}$$

and $x = (x_1, x_2, x_3) \geq 0$

For the given $f(x)$, Hessian matrix is

$$H^B = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Whose principal minors are $D_1 = [-2]$, $D_2 = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$

and $D_3 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ and $D_1 = |-2| = -2$

$D_2 = 4$, $D_3 = -8$

which are of alternate signs starting with $-$ve, so $f(x)$ is concave function (negative definite) of $x = (x_1, x_2, x_3)$

But as λ_i represents the rate of change of f w.r.t b_i, $\frac{\partial f}{\partial b_i} = \lambda_i$. Now as the right hand side of $h_i(\bar{x}) \le 0$ increases about zero, the solution space becomes less constraints $\Rightarrow$ f cannot decrease, so $\frac{\partial f}{\partial b_i} = \lambda_i \cancel{\le} 0$ i.e. $\lambda_i \ge 0$.

Thus when $h(x) < 0$, $\lambda = 0$ (Case I) and ~~when $\lambda > 0$~~, when $\lambda > 0$ $h(x) = 0$ (Case II)

Hence the Kuhn Tucker necessary condition is for the point $\bar{x}$ to be a ~~stationary point of~~ ~~maximum/stationary point~~ maximum for $f(\bar{x})$ subject to the constraint $\boxed{h_i(\bar{x}) = g_i(\bar{x}) - b_i \le 0}$ are as follows

$$\frac{\partial f(x)}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial h_i(x)}{\partial x_j} = 0 \quad \forall j = 1, 2, \dots n.$$

$$\lambda_i h_i(\bar{x}) = 0, \ \boxed{h_i(\bar{x}) \le 0} \text{ and } \lambda_i \ge 0, \ i = 1, 2, \dots, m.$$

Similarly the Kuhn-Tucker necessary conditions for the point $\bar{x}$ to be a point of ~~maximum~~ minimum for $f(x)$, subject to the constraints $\boxed{h_i(\bar{x}) = g_i(\bar{x}) - b_i \ge}$ are as follows

$$\frac{\partial f(x)}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial h_i(x)}{\partial x_j} = 0 \text{ for all } j = 1, 2, \dots n.$$

$$\lambda_i h_i(x) = 0, \ \boxed{h_i(\bar{x}) \ge 0} \text{ and } \lambda_i \ge 0 \ i = 1, \dots m$$

∂x_j $i=1$

$\lambda_i h_i(\bar{x}) = 0$, $h_i(\bar{x}) \le 0$ and ... sufficient if $f(x)$ and any ...

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Also $h_1(x)$, $h_2(x)$ are convex functions in x (8)

Thus the Kuhn-Tucker necessary conditions for max $g(x)$ are $\frac{\partial f(x)}{\partial x_j} = \sum_{i=1} \lambda_i \frac{\partial h_i(x)}{\partial x_j}$, $j = 1,2,3$.

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$\lambda_1 h_1(x) = 0$, $\lambda_2 h_2(x) = 0$, $h_1(x) \le 0$, $h_2(x) \le 0$

and $\lambda_1, \lambda_2 \geq 0$

i.e. the conditions are:

$j=1$: $-2x_1 + 4 = \lambda_1 + 2\lambda_2$ — (1)

$j=2$: $-2x_2 + 6 = \lambda_1 + 3\lambda_2$ — (2)

$-2x_3 = 0$ — (3)

$\lambda_1 (x_1 + x_2 - 2) = 0$ — (4)

$\lambda_2 (2x_1 + 3x_2 - 12) = 0$ — (5)

$x_1 + x_2 - 2 \le 0$ — (6)

$2x_1 + 3x_2 - 12 \le 0$ — (7)

$\lambda_1, \lambda_2 \geq 0$ — (8)

There are four different cases —

Case I: $\lambda_1 = 0$ and $\lambda_2 = 0$.

In this case from (1) & (2) we have.

$\cancel{-2x_1 + 4 = 2x_2 + 6 \Rightarrow 2x_2 + 2x_1 = 2}$

$\cancel{\text{or } x_1 + x_2 = 1}$ — (9)

$-2x_1 + 4 = 0 \therefore x_1 = -4/-2 = 2$.

& $-2x_2 + 6 = 0 \Rightarrow x_2 = 3$.

~~With~~ These value of $x_1 = 2$, $x_3 = 3$ does not satisfy (6) and (7) so this solution is discarded.

Case II $\lambda_1 \neq 0, \lambda_2 = 0$ 148

from (1) & (2) we get

$-2x_1 + 4 = \lambda_1$ and $-2x_2 + 6 = \lambda_1$

$\Rightarrow -2x_1 + 4 = -2x_2 + 6$

or ~~$x_1 + x_2 = 2$~~

$-2x_1 + 2x_2 = 2$

or $-x_1 + x_2 = 1$

or $\boxed{x_1 - x_2 + 1 = 0}$ — (9)

or

If $\lambda_1 \neq 0$ then from (4) $\lambda_1 (x_1 + x_2 - 2) = 0$

$\therefore x_1 + x_2 - 2 = 0$ or $x_1 + x_2 - 2 = 0$ — (10)

solving (9) & (10) we get

$x_1 - x_2 + 1 = 0$

$x_1 + x_2 - 2 = 0$

$2x_1 - 1 = 0 \Rightarrow x_1 = 1/2$

$\boxed{x_1 = 1/2, \; x_2 = 3/2}$ Putting in any one of them

$x_1 + x_2 - 2 = 0$

$1/2 + x_2 - 2 = 0$

$x_2 = 2 - 1/2 = 3/2$

These values of x_1 and x_2 satisfy (6) and (7)

For these values of x_1 and x_2 we have from (1) & (2)

$-2(1/2) + 4 = \lambda_1$ & $-2(3/2) + 6 = \lambda_1$

i.e. $\lambda_1 = 3 > 0$, $\lambda_2 = 0$.

For the solution $x_1 = 1/2, x_2 = 3/2, x_3 = 0$

$f(x) = Z = 17/2$

$\lambda_i h_i(\bar{x}) = 0$; $h_i(x) \leq 0$ ...

(151) Method (AAAK)

(149) (10) 74

Case-III when $\lambda_1 = 0$ & $\lambda_2 \neq 0$:—

In this case from (1) & (2) we have

from (1) $-2x_1 + 4 = 2\lambda_2 x_3$ or ~~$-x_1 + 2 = \lambda_2$~~

from (2) $-2x_2 + 6 = 3\lambda_2 x_2$ or ~~$-x_2 + 3 = \frac{3}{2}\lambda_2$~~

$\Rightarrow -6x_1 + \cancel{12} = -4x_2 + \cancel{12}$

$\Rightarrow -\cancel{6}^{3}x_1 + \cancel{4}^{2}x_2 = 0$

$\Rightarrow -3x_1 + 2x_2 = 0$

or $3x_1 - 2x_2 = 0$ —— (11)

If $\lambda_2 \neq 0$ then from (5) & (6) we get

$2x_1 + 3x_2 - 12 = 0$ —— (12)

Solving (11) (12) we get

$3x_1 - 2x_2 = 0 \;] \times 3$

$2x_1 + 3x_2 = 12 \;] \times 2$

$9x_1 - 6x_2 = 0$

$4x_1 + 6x_2 = 24$

$13x_1 = 24 \quad \therefore x_1 = 24/13$

Substituting in any of the eqn say

$3x_1 - 2x_2 = 0$

$\frac{3 \times \cancel{24}^{12}}{13} = \cancel{2}x_2$

$\boxed{x_1 = \frac{24}{13}}$; $\boxed{x_2 = \frac{26}{13}}$ Also from (13) $x_3 = 0$

These values does not satisfy eqn (6)

so this solution is discarded.

Case - IV When $\lambda_1 \neq 0$ & $\lambda_2 \neq 0$: 150

In this case from eqn (9) & (8) we have

$x_1 + x_2 = 2$ — (13) $2x_1 + 3x_2 = 12$ — (14).

$$2x_1 + 3x_2 = 12 \times 1$$
$$2x_1 + 2x_2 = 2 \times 2$$
$$\boxed{x_2 = 8}$$

→ from (1) $x_1 + x_2 = 2$

$x_1 + 8 = 2$

or $\boxed{x_1 = -6}$

As $x_1 = -6 < 0$ so this solution is also discarded.

Hence the optimal solution giving the max-value of $f(\bar{x})$ at $\bar{x} = (1/2, 3/2, 0)$ as Max value of $Z = 17/2$.

Ans

RKG.
Q12
P-84
Ex-21

Vaishali

Maximize $Z = x_1^2 + x_2^2 + x_3^2$

Subject to the constraints

$$x_1 + x_2 + 3x_3 = 2$$
$$5x_1 + 2x_2 + x_3 = 5$$

and $x_1, x_2, x_3 \geq 0$.

Problem associated

NLPP